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ON A VARIATIONAL INEQUALITY FOR THE HODOGRAPH METHOD

Robert A. Hummel

1. Planar fluid flow

The hodograph method has been studied extensively for incompressible and compressible inviscid irrotational planar fluid flow. The principal advantage of the hodograph method is that the system of differential equations become linear when expressed in terms of the hodograph variables. The major disadvantages are that the hodograph transformation may not be one-to-one, and that the boundaries of the hodograph domain are unknown. Recently, the use of variational inequalities has been shown to overcome these difficulties in certain cases. In this paper, we outline the current status of these results.

The equations of motion for planar, inviscid, irrotational fluid flow are

$$\operatorname{div} \rho(|\bar{q}|)\,\bar{q} = 0 \tag{1}$$

$$\operatorname{curl} \bar{q} = 0 \tag{2}$$

where $\rho = \rho(|q|)$ is a given positive decreasing function,

and $\bar{q} = \bar{q}(x,y) = (q_1(z), q_2(z))$, $z = x+iy$.

Equation (1) is the "equation of continuity", and expresses the physical property of conservation of matter. The second equation is unphysical, since it results from a theorem whose hypotheses include a perfect fluid model of the material. As a special case, incompressible fluid flow results when $\rho(q)$ is simply a positive constant, instead of a given decreasing function. The usual model for $\rho(q)$ for a compressible fluid is obtained from the Bernoilli

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relation (whose derivation uses (2)), $|q|^2/2 + P/\rho$ = const., and an adiabatic pressure-density law, such as $p = c\rho^\gamma$.

It is not hard to show that the system (1)-(2) is elliptic whenever $M = M(q) = q/c < 1$, where

$$c^2 = - \frac{\rho \cdot q}{\rho'(q)} \tag{3}$$

For incompressible flow, the system is always elliptic. For a general density-speed relation $\rho(q)$, $M < 1$ for sufficiently small q. For ideal fluids, one can show that M is increasing in q, and that there is exactly one critical speed q^* satisfying $M(q^*) = 1$. Accordingly, the system (1)-(2) is elliptic providing the solution $\bar{q}$ is subsonic, i.e., $|\bar{q}| < q^*$ everywhere.

The ellipticity of the system (1)-(2) is equivalent to the statement that

$$V(z) = q_1(z) - iq_2(z)$$

is quasi-conformal. Indeed, for the incompressible case, we may take $\rho \equiv 1$ without loss of generality, in which case (1) and (2) are simply the Cauchy-Riemann equations for $V(z)$. The hodograph transformation replaces the physical coordinate z of the flow region with the coordinate $\zeta = V(z)$ formed by the components of velocity. If the map $z \to V(z)$ is locally one-to-one at a point, then the flow solution near that point can be represented by the inverse map, which is a function of the hodograph variable $\zeta = q_1 - iq_2$. For subsonic flow, since $V(z)$ is quasi-conformal, the singularities where the hodograph transformation is not one-to-one are discrete and well behaved.

The hodograph method using variational inequalities, as investigated by Brezis and Stampacchia [1] can treat a variety of planar fluid flow problems. Most of the work has concentrated on exterior domains - flow past an obstacle with prescribed velocity at infinity. However, flow in a channel with an obstacle, flow through a Lavalle nozzle, and flow past an obstacle with cavitation can all be treated by similar methods [2,3,4] . To date, these methods always require a convexity condition limiting the kinds of geometry of the profile and walls that can be treated. For example, for flow past an obstacle, we require the obstacle to be strictly convex. We will also require that the solution satisfy a zero-circulation condition. Because of the perfect fluid assumption, flow past an obstacle exhibits no drag, and there is a one parameter family of solutions, parameterized by

$$\Gamma = \int_{\partial\Gamma} q_1 dx + q_2 dy \ ,$$

where Γ is the obstacle profile. Variational methods in conjunction with the hodograph transform have dealt only with the case $\Gamma = 0$. Finally, we note that hodograph methods almost necessarily are restricted to planar fluid flow situations, whether or not variational inequalities are employed.

2. Flow past an obstacle

We restrict our discussion to the problem of finding the flow past a convex profile with prescribed velocity at infinity. We are given,

(i) $P \subset \mathbb{C}$, bounded, strictly convex, $C^{2,\infty}$,

(ii) q_∞ , $0 < q_\infty << q^*$, and

(iii) $\rho = \rho(q)$, density-speed relation of an ideal fluid, or $\rho \equiv 1$.

Defining the flow region $G = \mathbb{C} \setminus P$, the flow problem can be stated as

Find $\bar{q} = (q_1, q_2) \in [C^2(G) \cap C^0(G)]^2$, such that

$$\operatorname{div} \rho(|q|)\cdot\bar{q} = 0 \quad \text{in } G \qquad (2.1)$$

$$\operatorname{curl} \bar{q} = 0 \quad \text{in } G \qquad (2.2)$$

$$\bar{q} \cdot \bar{n} = 0 \quad \text{on } G \ , \qquad (2.3)$$

$$\bar{n} = \text{normal to } \partial P,$$

$$\bar{q}(z) \to (q_\infty, 0) \text{ as } |z| \to \infty \tag{2.4}$$

and

$$\Gamma = \int_{\partial} q_1 dx + q_2 dy = 0 \tag{2.5}$$

Existence and uniqueness of a subsonic solution to (2.1-2.5) are known [5] . Variational methods using the hodoraph transform may lead to an independent existence theorem, but the principal motivation is the simple numerical implementation of a variational inequality.

Associated with the solution $\bar{q}$ are the potential and stream functions

$$\phi_x = q_1 \ , \ \phi_y = q_2$$

$$\Psi_x = -\rho q_2 \ , \ \Psi_y = \rho q_1 \ .$$

The stream function Ψ is constant on streamlines, so that we may assume that $\Psi \equiv 0$ on ∂P. Eliminating ϕ, the stream function satisfies the quasilinear equation

$$(1 - \frac{q_1^2}{c^2})\Psi_{xx} - 2\,\frac{q_1 q_2}{c^2}\,\Psi_{xy} + (1 - \frac{q_2^2}{c^2})\Psi_{yy} = 0$$

If Ψ is known, q can be recovered using $\text{grad}\Psi = (-\rho q_2, \rho q_1)$, and the fact that $\rho(q) \cdot q$ is an increasing, and therefore invertible function of q, for $q < q^*$.

3. <u>The Hodograph Domain</u>

Since the function V(z) is not globally one-to-one, some points in the hodograph domain may be multiply covered. A key idea in the extension of hodograph methods to non-symmetric flow is to view the hodograph domain as a Riemann surface, where points which are covered more than once are separated into multiple sheets. When the profile is convex, the Riemann surface can be described <u>a-priori</u>.

The description of the Riemann surface depends on the functions chosen to coordinatize the sheets. Of course, the coordinates (q_1, q_2) are the most obvious choice, although $(\rho(q)\cdot q_1,\ \rho(q)\cdot q_2)$ can also serve. In hodograph methods, it is common to use a polar-coordinate type representation (θ, σ), where

$$\theta = \arg(q_1 + iq_2) \tag{3.1}$$

$$\sigma = \int_{|q|}^{q^*} \frac{\rho(s)}{s}\, ds \ . \tag{3.2a}$$

For incompressible flow, the definition

$$\sigma = -\log |q| \tag{3.2b}$$

must be used in place of (3.2a). Note that $\sigma = +\infty$ corresponds to a stagnation point $q_1 - iq_2 = 0$, whereas for compressible flow, $\sigma = 0$ is the sonic speed.

To derive a description of the Riemann surface in the (q_1, q_2) variables, one first observes that by the classical existence theory, the solution $V(z)$ is branched over $z = \infty$, where $V(\infty) = q_\infty$. Further, one can show that $V(z) = 0$ at exactly two points, both located on the profile boundary. At a point z on the profile boundary, $\overline{V(z)}$ lies in the same direction as a tangent to ∂P at z. Thus $V(\partial P)$ consists of two closed curves, joined at the origin, and both enclosing q_∞ in their bounded interior regions. Consequently, the Riemann surface consists of two sheets, with a first order branch point at $\zeta = q_\infty$; each sheet is bounded by one of the two closed curves (see Figure 1). It is convenient to view the surface cut along the line $0 \leq \zeta < q$. The two sheets are joined along

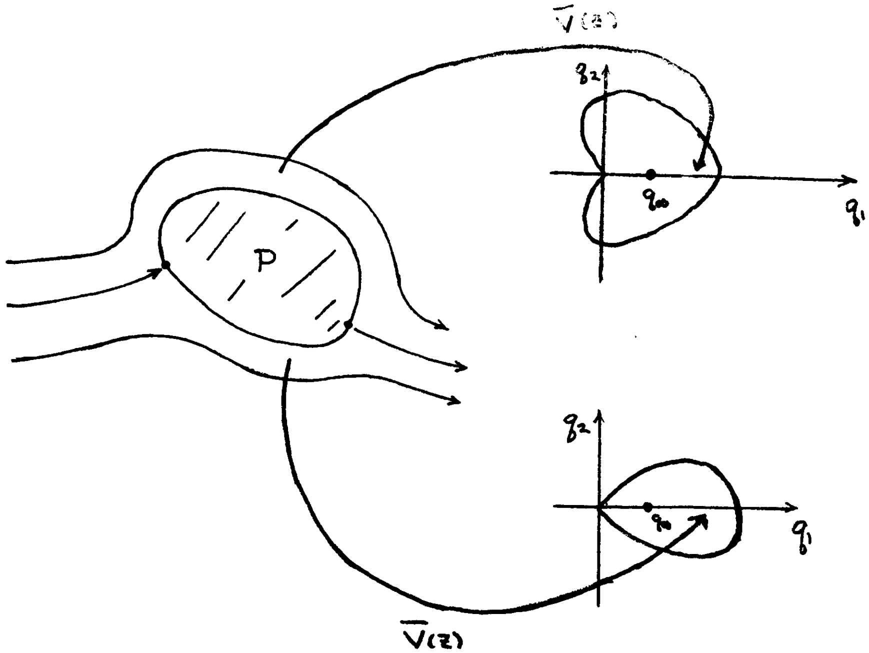

Figure 1.

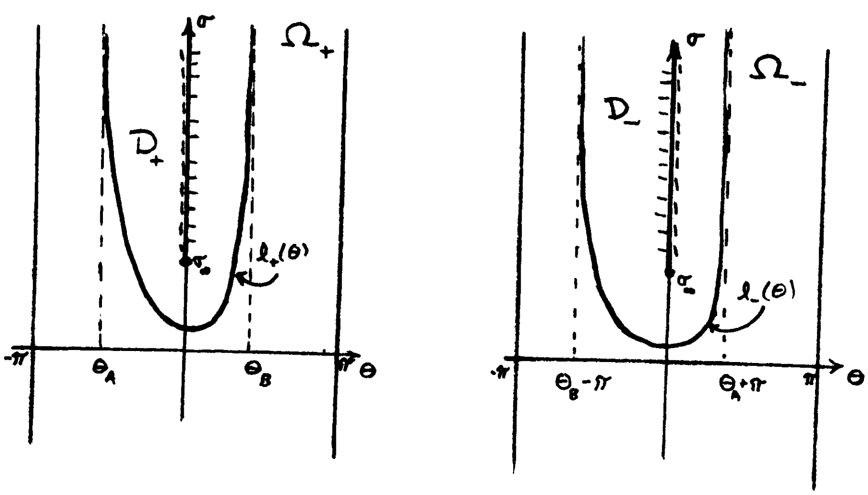

Figure 2.

the cut line in a criss-cross fashion, so that the upper shore on either sheet is identified with the lower shore on the other sheet.

In terms of the (θ,σ) variables, each sheet lies inside strip domains $\{(\theta,\sigma) = -\pi \leq \theta \leq \pi, \sigma \geq 0\}$. If we denote the two sheets by D_+ and D_-, we have

$$D_{\pm} = \{(\theta,\sigma) : \sigma \geq \ell_{\pm}(\theta)\} \setminus \{(0,\sigma) : \sigma \geq \sigma_{\infty}\} ,$$

where $\sigma_{\infty} = \sigma(q_{\infty})$ is the σ-value at the branch point. The curves $\ell_+(\theta)$ and $\ell_-(\theta)$ correspond to $\sigma(q)$ along the boundary of the profile in the physical plane, and constitute an unknown free boundary of the hodograph domain. Each curve has a left and right vertical assymptote at values of θ that correspond to the (unknown) locations of the stagnation points on ∂P. The hodograph domain is the Riemann surface D consisting of D_+ and D_-, branched over $(0,\sigma_{\infty})$, and identified along the cuts left shore to right and right shore to left (see Figure 2). By viewing the hodograph domain as a Riemann surface, we obtain a hodograph transform of $G \cup \{\infty\}$ onto D which is globally one-to-one.

Finally, D can be considered as a subset of a Riemann surface Ω, whose two sheets Ω_+ and Ω_- both consist of strip domains without the slits, with the same identifications on the slits as exist in D. The larger Riemann surface Ω will be the domain of the competing functions in the variational approach considered in the next section.

4. Flow equations in the hodograph variables

In the hodograph variables (θ,σ), Chapyglin's equation states that the stream function satisfies

$$k(\sigma)\frac{\partial^2\Psi}{\partial\theta^2} + \frac{\partial^2\Psi}{\partial\sigma^2} = 0$$

where

$$k(\sigma) = \frac{1-M^2}{\rho^2}$$

(See [6] for a derivation.)

For the variational method indicated below, we use the Legendre transform of Ψ , defined by

$$\overline{\Psi} = \Psi - x\,\frac{\partial\Psi}{\partial x} - y\,\frac{\partial\Psi}{\partial y}\ .$$

Note that $\overline{\Psi}$, as well as Ψ, can be considered to be defined in the hodograph domain D. It can be shown that Legendre transform $\overline{\Psi}$ satisfies the differential equation

$$\frac{1}{\rho^2}\ \frac{\partial^2\overline{\Psi}}{\partial\theta^2} + \frac{\partial}{\partial\sigma}\left(\frac{1}{k(\sigma)\rho^2}\ \frac{\partial\overline{\Psi}}{\partial\sigma}\right) = 0 \tag{4.1}$$

However, $\overline{\Psi}$ is non-zero on ∂D, and so we instead consider

$$U = \overline{\Psi} - q\rho(X_\pm(\theta)\ \sin\theta\ -\ Y_\pm(\theta)\ \cos\theta)\ , \tag{4.2}$$

where $(X_\pm(\theta),\ Y_\pm(\theta))$ is the coordinate of ∂P whose clockwise (resp, counter-clockwise) oriented tangent has argument θ . The function U, defined on D, does vanish on ∂D, and in fact satisfies $\mathrm{grad}U = 0$ on ∂D. We extend U to be zero in the strip domains Ω_+ and Ω_- outside D_+ and D_-. This function satisfies a variational inequality.

<u>Theorem</u>: Let $\mathbb{K}$ = {V defined on the Riemann surface Ω , denoted by $V_+(\theta,\sigma)$ on Ω_+, and $V_-(\theta,\sigma)$ on Ω_-, satisfying

i. $V_\pm \in H^1(\Omega_\pm)$

ii. $V_\pm(-\pi,\sigma) = V_\pm(\pi,\sigma) = 0 \qquad \forall\sigma$

iii. $\gamma V_+(0^+,\sigma) - \gamma V_-(0^-,\sigma) = q(\sigma)\cdot\rho(\sigma)\cdot H$ and
$\gamma V_+(0^-,\sigma) - \gamma V_+(0^+,\sigma) = q(\sigma)\cdot\rho(\sigma)\cdot H$, where γ is the trace operator, for $\sigma > \sigma_\infty$. (These are the "jump conditions" across the cuts.)

iv. $V_+ \geq 0$ on Ω_+, $V_- \leq 0$ on Ω_-}

Then U defined by (4.2) satisfies

$$U \varepsilon \mathbb{K}\ ,\ a(U,V-U) \geq \langle T,V-U\rangle \quad \text{for all} \quad V \varepsilon \mathbb{K}\ , \tag{4.3}$$

where $a(u,\zeta) = \iint_{\Omega_+ \cup \Omega_-} \frac{1}{\rho^2} u_\theta \zeta_\theta + \frac{1}{k\rho^2} u_\sigma \zeta_\sigma \, d\theta d\sigma$,

and
$$\langle T,\zeta\rangle = \iint_{\Omega_+ \cup \Omega} q/\rho \; R(\theta)\cdot\zeta(\theta,\sigma)d\theta d\sigma$$
$$+ \int_{\sigma_\infty}^{\infty} q/\rho\cdot W\cdot(\gamma\zeta_+(0^+,\sigma) - \gamma\zeta_+(0^-,\sigma))\, d\sigma \quad .$$

Here $H = Y_+(0) - Y_-(0)$, $W = X_+(0) - X_-(0)$,

and $R_\pm(\theta) = X_\pm'(\theta) \cos\theta + Y_\pm'(\theta) \sin\theta$.

Remarks: The proof of the variational inequality (4.3) is quite simple, and follows from (4.1), once it is shown that $U \varepsilon \mathbb{K}$. Of the conditions (i)-(iv) to be satisfied by a function in $\mathbb{K}$, only condition (iv) is difficult to verify to show that U belongs to $\mathbb{K}$. Verifying (iv) requires the introduction of a class of quasi-variational inequalities, and requires a monotonicity result. The details for the incompressible case are given in [7]. The compressible case is no more difficult.

The theorem is valid for both compressible and incompressible flow. In the incompressible case, one has $\rho \equiv 1$, $k(\sigma) \equiv 1$, and $q = e^{-\sigma}$. Note that in the compressible case, $\sigma \geq 0$ by definition, whereas for incompressible flow, an a-priori lower bound for σ also exists [1] , but may be negative. When the special condition of symmetry of the profile (with respect to the horizontal axis) is imposed, the variational inequality can be shown to reduce to the ones considered in earlier work.

5. Practical Consequences

Although the variational inequality (4.3) involves functions defined on a Riemann surface, and a fairly complicated distribution T, it nontheless leads to an extremely simple algorithm for finding flow solutions to the problem (2.1)-(2.5). The variational inequality may be solved numerically to yield U, or equivalently $\overline{\Psi}$, which is composed of two functions, Ψ_+ defined in D_+, and Ψ_- defined on D_-. Note that the subset $D \subseteq \Omega$ is determined from the solution U as the set where $U \neq 0$. Viewing $\overline{\Psi}_\pm$ as functions defined in $(w_1, w_2) = (\rho q_1, \rho q_2)$ variables, simple properties of the Legendre transform show that

$$\left(\frac{\partial}{\partial w_2} - i\,\frac{\partial}{\partial w_1}\right)\overline{\Psi}_\pm = x + iy = z \ ,$$

where $V(z) = q_1 - iq_2$. That is, from $\overline{\Psi}$ one can determine the physical point assigned to a given velocity.

To find $\overline{\Psi}$, the inequality (4.3) can be formulated in complementarity form. Namely, $\overline{\Psi}$ is a smooth function on Ω, satisfies the differential equation (4.1), and satisfies the constraints

$$\overline{\Psi}_+ \geq q\rho(X_+(\theta)\cdot\sin\theta - Y_+(\theta)\cdot\cos\theta) \quad \text{on } \Omega_+$$

$$\overline{\Psi}_- \leq q\rho(X_-(\theta)\cdot\sin\theta - Y_-(\theta)\cdot\cos\theta) \quad \text{on } \Omega_- \ .$$

One can regard the problem as a coupled system involving $\overline{\Psi}_+$ and $\overline{\Psi}_-$, with the coupling expressed as continuity across the identifed cuts.

The use of variational inequalities for the hodograph method seems to be a natural way of locating the unknown boundary of D. The Riemann surface appears in a natural fashion also, as the image of a hodograph transform which has been made one-to-one. The convexity requirement for P has been used in the representation of the boundary, and more importantly, to specify the general structure of the Riemann surface, so that the competing functions may be defined on a fixed Riemann surface Ω .

References

1. Brezis, H., and G. Stampacchia, "The hodograph method in fluid-dynamics in the light of variational inequalities," Arch. Rat. Mech. & Analysis 61, (1976), p. 1-18.
2. Tomarelli, F., "Un probléme de fluidynamique avec les inequations variationelles," C.R. Acad. Sci. Paris 286 (1978), p999-1002.
3. Shimborsky, E., "Variational methods applied to the study of symmetric flows in laval nozzles," (Preprint).
4. Brezis, H. and G. Duvaut, "Ecoulement avec sillage autour d'un profile symétrique sans incidence," C.R. Acad. Sci. Paris 276 (1973), p875-878.
5. Bers, L., "Existence and uniqueness of a subsonic flow past a given profile," C.P.A.M. 7 , (1954), p441-504.
6. Bers, L., Mathematical Aspects of Subsonic and Transonic Gas Dynamics, Chapman and Hall, 1958.
7. Hummel, R., "The hodograph method for convex profiles," submitted to Annali de Pisa, (preprint available from the author).